Ejercicios resueltos de integrales

1

$$\int x^2 \ln x \, dx$$

$$u = \ln x \xrightarrow{\text{derivar}} u' = \frac{1}{x}$$

$$v' = x^2 \xrightarrow{\text{integrar}} v = \frac{x^3}{3}$$

$$\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C =$$

$$= \frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right) + C$$

2

$$\int x^2 \operatorname{sen} 3x \, dx$$

$$u = x^2 \xrightarrow{\text{derivar}} u' = 2x$$

$$v' = \operatorname{sen} 3x \xrightarrow{\text{integrar}} v = -\frac{1}{3}\cos 3x$$

$$\int x^2 \operatorname{sen} 3x \, dx = -\frac{1}{3}x^2 \cos 3x + \frac{2}{3}\int x \cos 3x \, dx$$

$$u = x \xrightarrow{\text{derivar}} u' = 1$$

$$v' = \cos 3x \xrightarrow{\text{integrar}} v = \frac{1}{3}\operatorname{sen} 3x$$

$$\int x^2 \operatorname{sen} 3x \, dx = -\frac{1}{3}x^2 \cos 3x + \frac{2}{3}\left(\frac{1}{3} x \operatorname{sen} 3x - \frac{1}{3}\int \operatorname{sen} 3x \, dx\right) =$$

3

$$\int \text{arc sen}\, x\, dx$$

$$u = \text{arc sen}\, x \xrightarrow{\text{derivar}} u' = \frac{1}{\sqrt{1-x^2}}$$

$$v' = 1 \xrightarrow{\text{integrar}} v = x$$

$$\int \text{arc sen}\, x\, dx = x\,\text{arc sen}\, x + \int \frac{x}{\sqrt{1-x^2}}\, dx =$$

$$= x\,\text{arc sen}\, x + \sqrt{1-x^2} + C$$

4

$$\int \frac{\ln x}{x}\, dx$$

$$u = \ln x \xrightarrow{\text{derivar}} u' = \frac{1}{x}$$

$$v' = \frac{1}{x} \xrightarrow{\text{integrar}} v = \ln x$$

$$\int \frac{\ln x}{x}\, dx = \ln^2 x - \int \frac{\ln x}{x}\, dx$$

$$2\int \frac{\ln x}{x}\, dx = \ln^2 x$$

$$\int \frac{\ln x}{x}\, dx = \frac{1}{2}\ln^2 x + C$$

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$$\int \frac{x}{(x+1)(x^2+x+1)}\, dx$$

$$\frac{x}{(x+1)(x^2+x+1)} = \frac{A}{(x+1)} + \frac{Mx+N}{x^2+x+1}$$

$$\frac{x}{(x+1)(x^2+x+1)} = \frac{A(x^2+x+1)+(Mx+N)(x+1)}{(x+1)(x^2+x+1)}$$

$$x = (A+M)x^2 + (A+M+N)x + A+N$$

Igualamos los coeficientes de los dos miembros.

$$\begin{cases} 0 = A+M \\ 1 = A+M+N \\ 0 = A+N \end{cases} \qquad A=-1 \quad M=1 \quad N=1$$

$$\int \frac{x}{(x+1)(x^2+x+1)}dx = -\int \frac{dx}{x+1} + \int \frac{x+1}{x^2+x+1}dx =$$

La primera integral es de tipo logariítmico y la segunda la tenemos que descomponer en dos, que serán de **tipo logarítmico y tipo arcotangente**.

Multiplicamos por 2 en la segunda integral para ir preparádola.

$$-\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{2x+2}{x^2+x+1}dx =$$

El 2 del numerador de segunda integral lo transformamos en 1 + 1.

$$-\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}dx$$

Descomponemos la segunda integral en otras dos.

$$-\int \frac{dx}{x+1} + \frac{1}{2}\left(\int \frac{2x+1}{x^2+x+1}dx + \int \frac{1}{x^2+x+1}dx\right) =$$

$$-\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}dx + \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

Las dos primeras integrales son de tipo logarítmico.

$$= -\ln(x+1) + \frac{1}{2}\ln(x^2+x+1) + \frac{1}{2}\int \frac{1}{x^2+x+1}dx$$

La integral que nos queda es de tipo arcotangente.

Vamos a transformar el denominador de modo que podamos aplicar la fórmula de la integral del arcotangente.

Transformamos el denominador en un binomio al cuadrado.

$$\int \frac{1}{x^2+x+1}dx = \int \frac{1}{\left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}+1}dx = \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}dx =$$

Multiplicamos numerador y denominador por 4/3, para obtener uno en el denominador.

Dentro del binomio al cuadrado multiplicaremos por la raíz cuadrada de 4/3.

$$= \int \frac{\frac{4}{3}}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1}dx = \int \frac{\frac{2}{\sqrt{3}}\cdot\frac{2}{\sqrt{3}}}{\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2+1}dx =$$

$$= \frac{2}{\sqrt{3}}\int \frac{\frac{2}{\sqrt{3}}}{1+\left(\frac{2}{\sqrt{3}}\frac{2x+1}{2}\right)^2}dx = \frac{2}{\sqrt{3}}\int \frac{\frac{2}{\sqrt{3}}}{1+\left(\frac{2x+1}{\sqrt{3}}\right)^2}dx =$$

$$= \frac{2}{\sqrt{3}} \operatorname{arc\,tg} \frac{2x+1}{\sqrt{3}} + C$$

$$\int \frac{x}{(x+1)(x^2+x+1)}dx = -\ln(x+1) + \frac{1}{2}\ln\left(x^2+x+1\right) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}} \operatorname{arc\,tg} \frac{2x+1}{\sqrt{3}} + C =$$

$$= -\ln(x+1) + \frac{1}{2}\ln\left(x^2+x+1\right) + \frac{1}{\sqrt{3}} \operatorname{arc\,tg} \frac{2x+1}{\sqrt{3}} + C$$

6

$$\int \frac{3x-4}{x^2+2x+4}dx$$

Sumamos y restamos 3 en el numerador, descomponemos en dos fracciones y en la primera sacamos factor común 3.

$$\int \frac{3x+3-3-4}{x^2+2x+4}dx = 3\int \frac{x+1}{x^2+2x+4}dx - 7\int \frac{1}{x^2+2x+4}dx =$$

Multiplicamos y dividimos en la primera fracción por 2.

$$= \frac{3}{2}\int \frac{2x+2}{x^2+2x+4}dx - 7\int \frac{1}{x^2+2x+4}dx =$$

$$= \frac{3}{2}\ln\left(x^2+2x+4\right) - 7\int \frac{1}{x^2+2x+4}dx =$$

$$\int \frac{1}{x^2+2x+4}dx$$

Vamos a transformar el denominador de modo que podamos aplicar la fórmula de la integral del arcotangente.

Transformamos el denominador en un binomio al cuadrado.

$$x^2+2x+4 = x^2+2x+1+3 = (x+1)^2+3 = 3\left[\left(\frac{x+1}{\sqrt{3}}\right)^2+1\right]$$

$$\int \frac{1}{x^2+2x+4}dx = \frac{1}{3}\int \frac{dx}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1}$$

Realizamos un **cambio de variable**.

$$\frac{x+1}{\sqrt{3}} = t \qquad \frac{1}{\sqrt{3}}dx = dt$$

$$\frac{1}{3}\int \frac{dx}{\left(\frac{x+1}{\sqrt{3}}\right)^2+1} = \frac{1}{3}\int \frac{\sqrt{3}}{t^2+1}dt = \frac{\sqrt{3}}{3}\operatorname{arc\,tg} t + C = \frac{\sqrt{3}}{3}\operatorname{arc\,tg}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

$$\int \frac{3x-4}{x^2+2x+4}dx = \frac{3}{2}\ln\left(x^2+2x+4\right) - \frac{7\sqrt{3}}{3}\operatorname{arc\,tg}\left(\frac{x+1}{\sqrt{3}}\right) + C$$

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$$\int \frac{3^x}{1+3^x}\,dx$$

$3^x = t$

$3^x \ln 3\, dx = dt$ $\qquad$ $dx = \dfrac{dt}{t \cdot \ln 3}$

$$\int \frac{t}{1+t}\,\frac{dt}{t \cdot \ln 3} = \frac{1}{\ln 3}\int \frac{dt}{1+t} = \frac{1}{\ln 3}\ln(1+t) + C =$$

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$$\int \frac{dx}{\sqrt{4-x^2}}$$

$x = 2\,\text{sen}\, t$

$dx = 2\cos t\, dt$

$$\int \frac{2\cos t\, dt}{\sqrt{4-4\,\text{sen}^2 t}} = \int \frac{2\cos t\, dt}{2\cos t} = \int dt = t + C$$

$x = 2\,\text{sen}\, t$ $\qquad$ $t = \text{arc sen}\,\dfrac{x}{2}$

$$\int \frac{dx}{\sqrt{4-x^2}} = \text{arc sen}\,\frac{x}{2} + C$$

9

$$\int \sqrt{\frac{x+2}{x-1}}\,dx$$

$$\frac{x+2}{x-1}=t^2$$

$x+2=t^2x-t^2 \qquad t^2x-x=t^2+2 \qquad x\left(t^2-1\right)=t^2+2$

$$dx=\frac{-6t}{\left(t^2-1\right)^2}\,dt$$

Integramos por partes.

$$\int\sqrt{\frac{x+2}{x-1}}\,dx=\int\sqrt{t^2}\,\frac{-6t}{\left(t^2-1\right)^2}\,dt=\int\frac{-6t^2}{\left(t^2-1\right)^2}\,dt$$

$$u=t \quad \xrightarrow{\text{derivar}} \quad u'=1$$

$$v'=\frac{-6t}{\left(t^2-1\right)^2} \quad \xrightarrow{\text{integrar}} \quad v=\frac{3}{t^2-1}$$

$$\int\frac{-6t^2}{\left(t^2-1\right)^2}\,dt=\frac{3t}{t^2-1}-3\int\frac{dt}{t^2-1}$$

Se realiza la **integral racional**.

$$\frac{1}{t^2-1}=\frac{A}{t-1}+\frac{B}{t+1} \qquad 1=A(t+1)+B(t-1)$$

$$t=1 \qquad 1=2A \qquad A=\frac{1}{2}$$

$$\int\frac{-6t^2}{\left(t^2-1\right)^2}\,dt=\frac{3t}{t^2-1}-\frac{3}{2}\left(\int\frac{dt}{t-1}-\int\frac{dt}{t+1}\right)$$

$$\int\frac{-6t^2}{\left(t^2-1\right)^2}\,dt=\frac{3t}{t^2-1}-\frac{3}{2}\ln(t-1)+\frac{3}{2}\ln(t+1)+C$$

$$\frac{x+2}{x-1}=t^2 \qquad t=\sqrt{\frac{x+2}{x-1}}$$

$$\int\sqrt{\frac{x+2}{x-1}}\,dx=\frac{3\sqrt{\frac{x+2}{x-1}}}{\frac{x+2}{x-1}-1}-\frac{3}{2}\left[\ln\left(\sqrt{\frac{x+2}{x-1}}-1\right)-\ln\left(\sqrt{\frac{x+2}{x-1}}+1\right)\right]+C=$$

Aplicamos las **propiedades de los logaritmos**.

$$=(x-1)\sqrt{\frac{x+2}{x-1}}-\frac{3}{2}\ln\left(\frac{\sqrt{\frac{x+2}{x-1}}-1}{\sqrt{\frac{x+2}{x-1}}+1}\right)+C$$

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$$\int\frac{dx}{1+\operatorname{sen}x+\cos x}$$

$$t=\operatorname{tg}\frac{x}{2}\qquad\qquad dt=\frac{2dt}{1+t^2}$$

$$\int\frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}=\int\frac{2}{1+t^2+2t+1-t^2}\,dt=\int\frac{2}{2+2t}\,dt=$$

11

$$\int\frac{4e^{3x}}{1+e^{2x}}\,dx$$

$$e^x=t$$

$$e^x\,dx=dt\qquad\qquad dx=\frac{dt}{t}$$

$$\int\frac{4e^{3x}}{1+e^{2x}}\,dx=4\int\frac{t^3}{(1+t^2)t}\,dt=4\int\frac{t^2}{1+t^2}\,dt=4\int\frac{t^2+1-1}{1+t^2}\,dt=$$

$$4\left(\int dt-\int\frac{1}{1+t^2}\,dt\right)=4(t-\operatorname{arc\,tg}t)+C=$$

$$= 4\left(e^x - \text{arc tg}\, e^x\right) + C$$

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$$\int \frac{dx}{\sqrt{\text{sen}\, x \cos^3 x}}$$

$$\text{tg}\, x = t \qquad\qquad dx = \frac{dt}{1 + t^2}$$

$$\int \frac{\frac{dt}{1+t^2}}{\sqrt{\frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{(1+t^2)\sqrt{1+t^2}}}} = \int \frac{\frac{dt}{1+t^2}}{\sqrt{\frac{t}{(1+t^2)^2}}} = \int \frac{\frac{dt}{1+t^2}}{\frac{\sqrt{t}}{1+t^2}} =$$

$$= \int \frac{dt}{\sqrt{t}} = 2\int \frac{dt}{2\sqrt{t}} = 2\sqrt{t} + C = 2\sqrt{\text{tg}\, x} + C$$