Ejercicios resueltos de integrales

1

$$\int x \operatorname{sen} x \, dx$$

$$u = x \xrightarrow{\text{derivar}} u' = 1$$

$$v' = \operatorname{sen} x \xrightarrow{\text{integrar}} v = -\cos x$$

$$\int x \operatorname{sen} x \, dx = -x \cos x + \int \cos x \, dx =$$

$$= -x \cos x + \operatorname{sen} x + C$$

2

$$\int \frac{\ln x}{x^3} dx$$

$$u = \ln x \xrightarrow{\text{derivar}} u' = \frac{1}{x}$$

$$v' = \frac{1}{x^3} \xrightarrow{\text{integrar}} v = -\frac{1}{2x^2}$$

$$\int \frac{\ln x}{x^3} dx = -\frac{1}{2x^2} \ln x + \frac{1}{2} \int \frac{1}{x^3} dx = -\frac{1}{2x^2} \ln x - \frac{1}{4x^2} + C$$

3

$$\int (x^3 + 5x^2 - 2) e^{2x} dx$$

$$u = x^3 + 5x^2 - 2 \xrightarrow{\text{derivar}} u' = 3x^2 + 10x$$

$$v' = e^{2x} \xrightarrow{\text{integrar}} v = \frac{1}{2} e^{2x}$$

$$\int (x^3 + 5x^2 - 2) e^{2x} dx = \frac{1}{2} (x^3 + 5x^2 - 2) e^{2x} - \frac{1}{2} \int (3x^2 + 10x) e^{2x} dx =$$

$$u = 3x^2 + 10x \xrightarrow{\text{derivar}} u' = 6x + 10$$

$$v' = e^{2x} \xrightarrow{\text{integrar}} v = \frac{1}{2}e^{2x}$$

$$= \frac{1}{2}(x^3 + 5x^2 - 2)e^{2x} - \frac{1}{2}\left(\frac{1}{2}(3x^2 + 10x)e^{2x} - \frac{1}{2}\int (6x + 10)e^{2x}\,dx\right) =$$

$$u = 6x + 10 \xrightarrow{\text{derivar}} u' = 6$$

$$v' = e^{2x} \xrightarrow{\text{integrar}} v = \frac{1}{2}e^{2x}$$

$$= \frac{1}{2}(x^3 + 5x^2 - 2)e^{2x} - \frac{1}{4}(3x^2 + 10x)e^{2x} + \frac{1}{8}(6x + 10)e^{2x} + \frac{3}{4}\int e^{2x}dx =$$

$$= \frac{1}{2}(x^3 + 5x^2 - 2)e^{2x} - \frac{1}{4}(3x^2 + 10x)e^{2x} + \frac{1}{8}(6x + 10)e^{2x} - \frac{3}{8}e^{2x} + C =$$

$$= \left(\frac{1}{2}x^3 + \frac{7}{4}x^2 - \frac{7}{4}x - \frac{1}{8}\right)e^{2x} + C$$

5

$$\int e^x \cos x\, dx$$

$$\int e^x \cos x\, dx$$

$$u = e^x \xrightarrow{\text{derivar}} u' = e^x$$

$$v' = \cos x \xrightarrow{\text{integrar}} v = \operatorname{sen} x$$

$$\int e^x \cos x\, dx = e^x \operatorname{sen} x - \int e^x \operatorname{sen} x\, dx$$

$$u = e^x \xrightarrow{\text{derivar}} u' = e^x$$

$$v' = \operatorname{sen} x \xrightarrow{\text{integrar}} v = -\cos x$$

$$\int e^x \cos x\, dx = e^x \operatorname{sen} x + e^x \cos x - \int e^x \cos x\, dx$$

$$2\int e^x \cos x\, dx = e^x \operatorname{sen} x + e^x \cos x$$

$$\int e^x \cos x\; dx = \frac{1}{2} e^x (\operatorname{sen} x + \cos x) \; + C$$

6

$$\int \frac{3x^2 - 2x + 5}{(x+3)^3} dx$$

$$\frac{3x^2 - 2x + 5}{(x+3)^3} = \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{(x+3)^3}$$

$$3x^2 - 2x + 5 = A(x+3)^2 + B(x+3) + C$$

Para calcular A, B y C, sustituimos x por −3:

$x = -3$ $\qquad$ $38 = C$

Derivamos y volvemos a sustituir por −3:

$$6x - 2 = 2A(x+3) + B$$

$x = -3$ $\qquad$ $-20 = B$

Volvemos a derivar:

$6 = 2A$ $\qquad$ $A = 3$

$$\int \frac{3x^2 - 2x + 5}{(x+3)^3} dx = \int \frac{3}{x+3} dx - \int \frac{20}{(x+3)^2} dx + \int \frac{38}{(x+3)^3} dx =$$

$$= 3 \ln(x+3) + \frac{20}{x+3} - \frac{19}{(x+3)^2} + C$$

También podemos hallar los coeficientes realizando las operaciones e igualando coeficientes:

$$3x^2 - 2x + 5 = Ax^2 + (6A + B)x + 9A + 3B + C$$

$$\begin{cases} & 3 = A \\ -2 = 18 + B & B = -20 \\ 5 = 27 + C & C = 38 \end{cases}$$

7

$$\int x\sqrt{1+x}\,dx$$

$$1 + x = t^2 \qquad\qquad x = t^2 - 1$$

$$dx = 2t\,dt$$

$$\int (t^2 - 1)\cdot t \cdot 2t\,dt = \int (2t^4 - 2t^2)\,dt = \frac{2}{5}t^5 - \frac{2}{3}t^3 + C$$

$$t = \sqrt{1+x}$$

$$\frac{2}{5}\left(\sqrt{1+x}\right)^5 - \frac{2}{3}\left(\sqrt{1+x}\right)^3 + C =$$

$$= \frac{2}{5}(1+x)^2\sqrt{1+x} - \frac{2}{3}(1+x)\sqrt{1+x} + C$$

8

$$\int \frac{dx}{\sqrt{1+e^x}}$$

$$1 + e^x = t^2 \qquad\qquad e^x = t^2 - 1$$

$$e^x\,dx = 2t\,dt \qquad\qquad dx = \frac{2t\,dt}{t^2 - 1}$$

$$\int \frac{2t\,dt}{(t^2-1)t} = 2\int \frac{dt}{t^2-1}$$

$$\frac{1}{t^2-1} = \frac{A}{t+1} + \frac{B}{t-1} \qquad 1 = A(t-1) + B(t+1)$$

$t = -1 \qquad 1 = -2A \qquad A = -\frac{1}{2}$

$t = 1 \qquad 1 = 2B \qquad B = \frac{1}{2}$

$$2\int \frac{dt}{t^2-1} = -\int \frac{dt}{t+1} + \int \frac{dt}{t-1} = -\ln(t+1) + \ln(t-1) + C = \ln\left(\frac{t-1}{t+1}\right) + C$$

$t = \sqrt{1+e^x}$

$$\int \frac{dx}{\sqrt{1+e^x}} = \ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + C$$

8

$$\int \frac{dx}{\sqrt{1+e^x}}$$

$1 + e^x = t^2 \qquad e^x = t^2 - 1$

$e^x\,dx = 2t\,dt \qquad dx = \frac{2t\,dt}{t^2-1}$

$$\int \frac{2t\,dt}{(t^2-1)t} = 2\int \frac{dt}{t^2-1}$$

$$\frac{1}{t^2-1} = \frac{A}{t+1} + \frac{B}{t-1} \qquad 1 = A(t-1) + B(t+1)$$

$t = -1 \qquad 1 = -2A \qquad A = -\frac{1}{2}$

$t = 1 \qquad 1 = 2B \qquad B = \frac{1}{2}$

$$2\int \frac{dt}{t^2-1} = -\int \frac{dt}{t+1} + \int \frac{dt}{t-1} = -\ln(t+1) + \ln(t-1) + C = \ln\left(\frac{t-1}{t+1}\right) + C$$

$$t = \sqrt{1+e^x}$$

$$\int \frac{dx}{\sqrt{1+e^x}} = \ln\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right) + C$$

9

$$\int \frac{dx}{x\sqrt{x^2-2}}$$

$$x = \sqrt{2}\sec t$$

$$dx = \sqrt{2}\sec t \,\mathrm{tg}\, t\, dt$$

$$\int \frac{\sqrt{2}\sec t\,\mathrm{tg}\,t}{\sqrt{2}\sec t\sqrt{2\sec^2 t-2}}dt = \int \frac{\mathrm{tg}\,t}{\sqrt{2\left(\sec^2 t-1\right)}}dt = \int \frac{\mathrm{tg}\,t}{\sqrt{2}\,\mathrm{tg}\,t}dt =$$

$$= \frac{1}{\sqrt{2}}\int dt = \frac{1}{\sqrt{2}}\, t + C$$

$$x = \sqrt{2}\sec t \qquad x = \frac{\sqrt{2}}{\cos t} \qquad \cos t = \frac{\sqrt{2}}{x} \qquad t = \arccos\left(\frac{\sqrt{2}}{x}\right)$$

$$\int \frac{dx}{x\sqrt{x^2-2}} = \frac{1}{\sqrt{2}}\arccos\left(\frac{\sqrt{2}}{x}\right) + C$$

10

$$\int \frac{\sqrt{x+1}+2}{\sqrt[3]{(x+1)^2}-\sqrt{x+1}}dx$$

$$x + 1 = t^6$$

$dx = 6t^5\,dt$

$$\int \frac{t^3+2}{t^4-t^3}\,6t^5\,dt = 6\int \frac{t^5+2t^2}{t-1}\,dt = 6\int \left(t^4+t^3+t^2+3t+\frac{3}{t-1}\right)dt =$$

$$= \frac{6}{5}t^5 + \frac{3}{2}t^4 + 2t^3 + 9t^2 + 18\ln(t-1) + C$$

$x+1 = t^6$ $\qquad\qquad$ $t = \sqrt[6]{x+1}$

$$= \frac{6}{5}\sqrt[6]{(x+1)^5} + \frac{3}{2}\sqrt[6]{(x+1)^4} + 2\sqrt[6]{(x+1)^3} + 9\sqrt[6]{(x+1)^2} + 18\ln\left(\sqrt[6]{(x+1)} - 1\right) + C =$$

$$= \frac{6}{5}\sqrt[6]{(x+1)^5} + \frac{3}{2}\sqrt[3]{(x+1)^2} + 2\sqrt{x+1} + 9\sqrt[3]{x+1} + 18\ln\left(\sqrt[6]{(x+1)} - 1\right) + C$$

11

$$\int \operatorname{cosec}^3 x\,dx$$

$$\int \operatorname{cosec}^3 x\,dx = \int \frac{dx}{\operatorname{sen}^3 x}$$

$\operatorname{tg}\frac{x}{2} = t$ $\qquad\qquad$ $dx = \frac{2dt}{1+t^2}$

$$\int \frac{1}{\left(\frac{2t}{1+t^2}\right)^3}\,\frac{2dt}{1+t^2} = \int \frac{(1+t^2)^2}{4t^3}\,dx = \int \frac{1+2t^2+t^4}{4t^3}\,dx =$$

$$= \frac{1}{4}\int \frac{dt}{t^3} + \frac{1}{2}\int \frac{dt}{t} + \frac{1}{4}\int t\,dt = -\frac{1}{8t^2} + \frac{1}{2}\ln t + \frac{1}{8}t^2 + C =$$

12

$$\int \frac{e^{4x}+3}{e^{3x}}\,dx$$

$e^x = t$

$e^x dx = dt \qquad dx = \frac{dt}{t}$

$$\int \frac{e^{4x}+3}{e^{3x}}dx = \int \frac{t^4+3}{t^3 \cdot t}dt = \int \frac{t^4+3}{t^4}dt = \int dt + 3\int \frac{dt}{t^4} = t - \frac{1}{t^3} + C =$$

$$= e^x - \frac{1}{e^{3x}} + C$$